NON-LINEAR NATURAL FREQUENCIES OF A ROTATING BEAM MOUNTED ON AN ELASTIC FOUNDATION

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Authorization Form

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Dedication

To

My family

Acknowledgments

I would like to sincerely and wholeheartedly thank Dr.Ahmad AL-Qaisia and Dr. Basem Al-Bedoor for their guidance and kindness throughout this work and their valuable suggestions and help.

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a_1 Hub mass constant.

a_2 Tip mass constant.

a_3 Rotary inertia constant.

a_4 Torsional root spring constant.

a_5 Dimensionless linear spring constant in X direction.

a_6 Dimensionless Linear spring constant in Y direction.

AMM Assumed Modes Method.

A_θ Rotational transformation matrix from the xy system to XY system.

A_η Rotational transformation matrix from the $x'y'$ to xy system.

E Modulus of elasticity of the beam.

I Area moment of inertia of the beam

I_h Mass moment of inertia of the hub

J_t Rotary inertia of the tip mass.

K_x Spring constant in the X direction.

K_y Spring constant in the Y direction.

K_T Torsional spring constant at beam root.

L Lagrangian.

l Length of the beam.

m_b Mass of the beam.

m_h Mass of the hub.

m_t Tip mass.

MMS Method of Multiple Scales.

MHB	Method of Harmonic Balance.
q	Unknown time modulation of the assumed deflection mode.
R	Radius of the hub.
R_h	Position vector of the origin of the *xy* system in the inertial reference frame XY
R_o	Position vector of the hub center in the inertial reference frame *XY*.
R_p	Global position of an arbitrary point p on the flexible beam.
r_p	Position vector of the point p in the $x'y'$.
s	Axial position of the point p measured relative to the *x'y'* system.
T	Total kinetic energy of the system.
T_b	Kinetic energy of the beam.
T_h	Kinetic energy of the hub.
T_t	Kinetic energy of the tip mass.
t	Time.
t^*	Dimensionless time.
u	Axial displacement of beam.
V	Total potential energy of the system.
V_A	potential energy of the axial shortening due to transverse deformation and the motion generated inertial force
V_b	potential energy of the beam.
V_E	The elastic potential energy of the beam due to bending.
	Potential energy of the hub .
V_h	
V_t	Potential energy of the torsional spring..
v	Lateral displacement of the beam due to bending.
XY	Inertial reference frame.

xy	System of orthogonal axes rotating with the hub with the origin at the beam root.
$x'y'$	Principal axes.
β_1- β_{16}	Lagrangian coefficients.
$\dot{\gamma}$	Time derivative of the slope of the deformed beam.
ε	Small gauge parameter.
ε_0- ε_5	Equation of motion coefficients.
ς	Dimensionless axial position of the point p measured relative to the $x'y'$ system.
η	Attachment angle.
θ	Angular position of the hub.
$\dot{\theta}$	Rotating speed.
κ	Curvature.
Λ	Dimensionless excitation amplitude.
ρ	Mass of the beam per unit length.
ϕ	Assumed deflection mode.
Ω	Dimensionless rotating speed.
ω	Dimensionless non-linear natural frequency.
ω_e	Dimensionless excitation frequency.
ω_n	Natural frequency for non-rotating beam.

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ABSTRACT

In this study, the non-linear dynamic model, of a rotating beam with a tip mass and a flexible root, attached to a hub with an attachment angle which is mounted on an elastic foundation, is derived using the Lagrangian dynamics in conjunction with Assumed Modes Method. The Euler-Bernoulli beam theory is adapted. The model equations are coupled and non-linear.

The Method of Multiple Scales, and the Method of Harmonic Balance are used to study the effects of the hub radius, attachment angle, tip mass, rotary inertia, torsional spring constant, and non-dimensional rotating speed on the first three non-linear natural frequencies, the results showed that as the tip mass and rotary inertia increase, the non-linear natural frequencies decrease, while as the attachment angle, hub radius, and torsional spring constant increase the non-linear natural frequencies increase.

The results of this work have applications in various fields such as; space application, robotic manipulators and turbo-machinery blades, due to their importance in design, control, precise positioning and performance evaluation.

Introduction

In general all bodies that possess mass and elasticity can vibrate and accordingly most structures and mechanical systems can experience some kinds of vibration. This can be set in a variety of contexts ranging from a simple mass-spring system or gyroscopic instruments to helicopter, turbine, compressor and propeller blades.

The vibrational theory is concerned with the study of oscillatory motions of bodies and the forces associated with them. These studies constitute an important part of the structural and mechanical system design.

Oscillatory motion is one of the main reasons for several mechanical failures. In spite of great progress in mechanical design, engineers have not yet been able to fully prevent failures that result from vibration of moving or rotating elements in machines like turbines and compressor. Unwanted vibrations may lead to rapid wear of machine parts causing damages or excessive noises. Vibration must be studies and controlled properly. In spite of its undesirable effect vibration can be found useful in several industrial or consumer applications.

There has in recent years been an upsurge of interest in the vibrational analysis of elastic structures rotting at a constant angular velocity. Numerous structural configuration such as turbine, compressor, helicopter blades, spinning spacecraft, satellites, and also rotating shafts and linkages fall into this category.

The essential feature that distinguished such systems form non-rotating ones is in general, the complexity of the accelerations, which act throughout the system, in addition to the acceleration resulting form elastic structural deformations. The equations of motion may involve significant gyroscopic or Coriolis and centripetal accelerations. Also the stiffness characteristics of the structures may be modified by the steady state inertia loads induced by the centrifugal forces. Consistent mathematical models that addressed the beam as typical rotating element and considering only the beam transverse deflection have resulted in softening effect. This behavior contradicts the practical physics of the problem that forced researchers to seek techniques to compensate for this contradiction through studying the effect produced by the centrifugal force.

1.1 Problem Description

In this study, the equations of motion governing the nonlinear vibrations of isotropic, inextensible, rotating Euler–Bernoulli beam (with solid cross sectional area) with a tip mass and a flexible root, attached to a hub with an attachment angle which is mounted on an elastic foundation by using Lagrangian Dynamics in conjunction with the Assumed Modes Method (AMM).

The position vector of a typical material point is used in deriving the kinetic energy expression which includes the base two degrees of freedom, the rigid body rotation of the hub-beam system, and the tip mass. The geometrically exact curvature is employed in expressing the beam elastic potential energy due to the axial shortening. The system Lagrangian in conjunction with the assumed modes method, and after imposing the beam inextensibility constraint, is used to develop four degrees of freedom system. The four degrees of freedom being the horizontal and vertical position of the hub-beam assembly, the rigid body rotation of the hub, and the transverse deflection of the beam in the modal domain. First order multiple scales expansion, single term harmonic balance, and two term harmonic are used in studying the effect of root radius beam length-ratio torsional spring constant attachment angle, tip mass , and rotary inertia on non-linear natural frequencies.

1.2 Thesis Objectives

Deriving the equations of motion for a rotating beam with tip mass and flexible root attached to a hub mounted on an elastic foundation.

Solve the equation of motions using the Method of Multiple Scale and the Method of Harmonic Balance.

Study the effect of different parameters tip mass, rotary inertia, attachment angle and hub radius on non-linear natural frequency.

Literature Review

The vibration of a rotating beam has been studied earlier as Shilhansi (1958) and Prudli (1972) have shown that the rotation speed strengthens the beam and produces high natural frequencies. Kaza and Kavternik (1977) reported results on the non-linear flap-lag axial equations of a rotating beam.

Later Hoa (1977) used a finite element method based on a third order polynomial to investigate the vibration frequency of a rotating beam with a tip mass taken into account the effects of root radius and the setting angle. Bhat (1986) used the beam characteristic orthogonal polynomial in a Rayleigh-Ritz analysis to determine the natural frequencies of flexural modes of rotating cantilever beam with a tip mass at the free end. He concluded that the use of beam characteristic orthogonal polynomials in Rayleigh-Ritz method gives good results compared to other methods of solution.

Lee (1994) investigated the effect of gravity on the stability of a rotating cantilever beam in a vertical plane with constant angular velocity.

Surace *et al.* (1997) have used the Houblot and Brooks equations for non-uniform pretwisted rotting blade. And carryout the flap-lag torsional vibration analysis using an integral approach based on the use of Green functions, they have found that the dynamic characteristics obtained with this approach are in good agreement with the results of other methods.

Yoo and Shin (1998) derive the equations of motion of coupled stretching and bending rotating cantilever beam, then they used the modal analysis to solve the equation of motion. They have found that the non-linear natural frequency shows a significant difference compared to the results ignoring the coupling effect.

Fung and Yau (1999) studied the vibration of an arm rotates horizontally about clamped axis while other end is constrained to move against a curve. They ignored rotary inertia and shear deformation and derived the equations of motion and boundary conditions by using Hamilton's principle then they used a power series method to solve the equation of motion. Al-Beddor (2001) developed a reduced non-linear dynamic model for the shaft-disk blade unit to study effect of shaft torsional vibration on blade bending vibration.

Yoo *et al.* (2001) derived the equations of motion for the vibration analysis of rotating pre-twisted blade by using modeling method, which employs hybrid deformation variables, and then they investigated the effect of pre-twist angle and hub radius on tuned angular speed. They concluded that as hub radius and pre-twist angle increase tuned angular speed increase, but the hub radius affects more significantly than the pre-twist angle.

Vibration of rotating Timoshenko beam and the effect of Coriolis force on the natural frequencies was investigated by (Lin and Hasio ,2001). The equations of motion are derived by the d'Alembert principle and the virtual work principle. Then they expressed a solution for the equation of motion as a power series, they concluded that the Coriolis force effect on the natural frequencies might be neglected. Modeling of a rotating flexible inextensible arm mounted on a rigid hub with a setting angle is considered by (Hamden and Al-Bedoor ,2001), they have taken into the account the effect of axial shortening due to bending in the formulation of both the kinetic and potential energy , they concluded that the non-linear terms could improve the stability of the system dynamic behavior.

Al-Qaisia (2002) studied the variation of the natural frequency with vibration amplitude of a beam clamped with an angle to a rigid rotating hub and carrying a tip mass with rotary inertia.

Tsai (2004) studied the vibration behavior of full cycle of 60 blades. First he collect the geometric dimensions and material properties of a single blade, then he created a finite element model of a single blade and full cycle of 60 blades and performed the vibration analyses using ANSYS software.

Al-Qaisia (2004) developed a model of a rotating beam based on the large deformation theory using inextensibility condition and simulated it numerically, and studied the effect of; rotational speed, torque period, and beam length on the dynamic behavior of the rotating beam.

Recently Al-Qaisia and Al-Bedoor (2005) developed four different methods for accounting for axial shortening in rotating beam and used the method of time transformation to find the frequency for the different method. They have found that the Potential Energy Method (PM) give good results compared to other methods investigated.

The objectives of this work, is extend the analysis presented in (Al-Qaisia,2002 and 2004). And study the effect of base and root flexibility on non-linear natural frequencies. It can also be seen that the method of multiple scales and the method of harmonic balance used in this study haven't been used in studying the non-linear natural frequencies.

The interest of the model is that it can model many physical applications, for example; a turbo machinery blade clamped to a disk with an attachment angle, the bearing can be modeled as an elastic foundation and a flexible arm or a robotic manipulator carrying an end effector, which can be modeled as a flexible beam having a concentrated mass with rotary inertia at the free end.

Methods and Procedures

In this chapter, the equations of motion governing the nonlinear vibrations of a rotating beam with a tip mass and a flexible root are derived.

Various methods have been developed for deriving the equations of motion of rotating beam, here the Lagrangian approach in conjunction with the Assumed Modes Method (AMM) is followed. In particular, the approach used by (Al-Qaisia and Al-Bedoor, 2005) has been adopted.

3.1 Model Description and Assumptions

Figure (1) shows a uniform beam, having rectangular cross-sectional area, flexural rigidity (*EI*), constant length (*l*), and mass per unit length (ρ), with a concentrated tip mass with mass (m_t), and rotary inertia (J_t) attached to a hub of radius (*R*) with an angle (η), which is mounted on an elastic foundation, and rotates with constant angular velocity ($\dot{\theta}$).

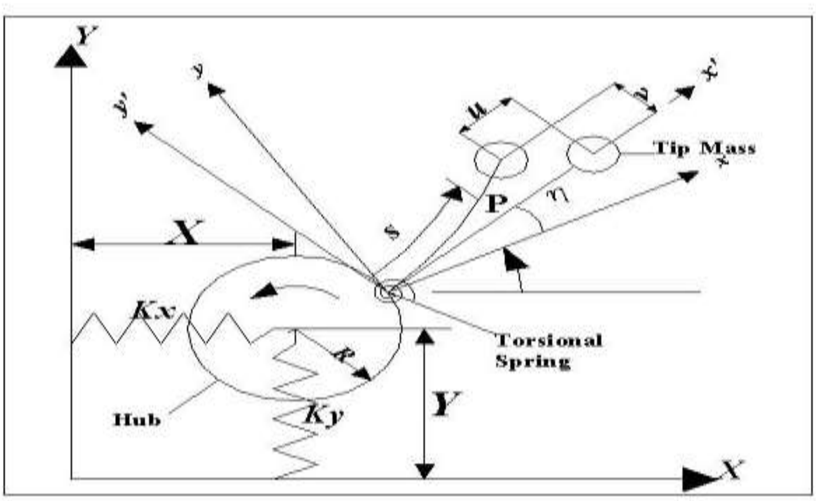

Figure (1) Rotating hub-beam system mounted on an elastic foundation

The coordinate systems used in developing the model are shown in figure (1), where *XY* denotes the inertial reference frame that is fixed in space which has the unit vectors I and J, while xy is the system of orthogonal axes rotating with the hub with the origin O at the root of the beam which has the unit vectors i and j, and $x'y'$ is a system of orthogonal axes with common origin O with xyz system and with orientations along the major dimensions of the beam: i.e., the x' axis is along the length of the beam, the y' axis is along the thickness of the beam, (also called the principal axes), which has the unit vectors i', and j'.

The beam deflection is assumed to be restricted to the horizontal x-y plane, the beam is assumed to be inextensible, composed of a homogeneous, isotropic material, and has a slender shape so that shear and rotary inertia effects are negligible. The hub is assumed to be a rigid uniform disk. The flexible root is assumed to be a torsional spring at the beam root. The elastic foundation is assumed to be linear springs with stiffness (K_x), and (K_y) in X and Y directions respectively

3 .2 Problem Formulation

In this section equations of motion will be derived using the Lagrangian approach in conjunction with the (AMM). First an expression for the position vector for a material point on the beam is found, then the kinetic energy expression for the system, non-linear curvature for the beam, the Inextensibility condition for the beam and the potential energy expression for the system is derived. After that the kinetic and potential energy expressions in conjunction with (AMM) is used to obtain the Lagrangian of the system. Then the equations of motion are derived using Lagrange Equation. Lastly they are simplified and written them in non-dimensional form

3.2.1 Position vector

In the analysis presented in this section, the beam model shown in figure (1) is used.

The global position vector of an arbitrary point P (R_p) in the inertial reference frame, on the flexible beam can be expressed as (Abed, 2003).

$$R_p = R_o + R_h + [A_\theta][A_\eta]r_p \qquad (3.1)$$

Where (R_o) is the position vector of the hub center in the inertial reference frame *XY* and can be represented as follows (Al-Bedoor *et al.*, 2002)

$$R_o = XI + YJ \qquad (3.2)$$

Where *X* and *Y* are respectively, the X and Y inertial components of the vector R_o.

(R_h) is the position vector of the origin of the *xy* system in the inertial reference frame *XY* and can be represented as follows (Al-Bedoor *et al.*, 2002).

$$R_h = R\cos\theta I + R\sin\theta J \qquad (3.3)$$

Where (θ) measures the angular position of the hub.

[A_θ] is the rotational transformation matrix from the *xy* system to *XY* system and is given by (Al-Bedoor *et al.*, 2002)

$$[A_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad (3.4)$$

[A_η] is the rotational transformation matrix form the $x'y'$ system to *xy* system and is given by (Abed, 2003)

$$[A_\eta]=\begin{bmatrix}\cos\eta & -\sin\eta\\ \sin\eta & \cos\eta\end{bmatrix}$$

(3.5)

(r_p) is the position vector of the point p in the $x'y'$ system and can be written in the form

$$r_p = si' + v(s,t)j'$$

(3.6)

Where (s) is the axial position of point p measured relative to the $x'y'$, and $v(s,t)$ is the lateral displacement of the point p along the y' axis.

The velocity of the arbitrary point p in the inertial reference frame can be obtained by differentiating equation (3.1) with respect to time (t) as follows:

$$\dot{R}_p = \dot{R}_o + \dot{R}_h + [A_\theta][A_\eta]\dot{r}_p + \frac{d}{dt}[A_\theta][A_\eta]r_p \quad (3.7)$$

Where dots denote differentiation with respect to time (t)

Substituting equations (3.2)-(3.6) and their derivatives into equation (3.7), the velocity vector of the point P is given by

$$\dot{R}_p =$$

$$[\dot{X} - (\dot{\theta}(R + s\cos\eta - v\sin\eta) + \dot{v}\cos\eta)\sin\theta + \left(-\dot{\theta}(s\sin\eta + v\cos\eta) - \dot{v}\sin\eta\right)\cos\theta]I$$

$$+[\dot{Y} + \left(\dot{\theta}(R + s\cos\eta - v\sin\eta) + \dot{v}\cos\eta\right)\cos\theta + \left(-\dot{\theta}(s\sin\eta + v\cos\eta) - \dot{v}\sin\eta\right)\sin\theta]J$$

(3.8)

Let

$$\alpha = \dot{\theta}(R + s\cos\eta - v\sin\eta) + \dot{v}\cos\eta \qquad (3.9)$$

And

$$\beta = -\dot{\theta}(s\sin\eta + v\cos\eta) - \dot{v}\sin\eta \qquad (3.10)$$

Then (3.8) can be written as

$$\dot{R}_p = \lfloor \dot{X} - \alpha \sin\theta + \beta \cos\theta \rfloor I + \lfloor \dot{Y} + \alpha \cos\theta + \beta \sin\theta \rfloor J \qquad (3.11)$$

3.2.2. System Kinetic Energy

The kinetic energy of the system (T) consists of three parts, the kinetic energy of the hub (T_h), the kinetic energy of the beam (T_b), and the kinetic energy of the tip mass (T_t).

$$T = T_h + T_b + T_t \qquad (3.12)$$

so that the kinetic energy of the hub can be expressed as follows

$$T_h = \frac{1}{2} m_h \dot{X}^2 + \frac{1}{2} m_h \dot{Y}^2 + \frac{1}{2} I_h \dot{\theta}^2 \qquad (3.13)$$

Where (m_h) is the mass of the hub, and (I_h) is the mass moment of inertia of the hub and is given by

$$I_h = \frac{1}{2} m_h R^2 \qquad (3.14)$$

Substitute (3.14) into (3.13) we get

$$T_h = \frac{1}{2} m_h \dot{X}^2 + \frac{1}{2} m_h \dot{Y}^2 + \frac{1}{4} m_h R^2 \dot{\theta}^2 \qquad (3.15)$$

The kinetic energy of the beam is given by (Al-Qaisia Al-Bedoor, 2005)

$$T_b = \frac{1}{2} \rho \int_0^l \dot{R}_p^T . \dot{R}_p \, ds \qquad (3.16)$$

Substituting Eq. (3.11) into (3.16) then the kinetic energy of the beam can be written as

$$T_b = \frac{1}{2} \rho \int_0^l \left(\dot{X}^2 + \dot{Y}^2 + \alpha^2 + \beta^2 \right) ds \qquad (3.17)$$

The kinetic energy of the tip mass is given by (Al-Qaisia, 2004)

$$T_t = \frac{1}{2} m_t \dot{R}_p^T \dot{R}_p \Big|_{s=l} + \frac{1}{2} J_t \left(\dot{\theta} + \dot{\gamma}\big|_{s=l} \right)^2 \qquad (3.18)$$

Where ($\dot{\gamma}$) is the time derivative of the slope of the deformed beam and is given by (Al-Qaisia, 2004)

$$\dot{\gamma} = \dot{v}' \left(1 + \frac{1}{2} v'^2 \right) \qquad (3.19)$$

Substitute Eqs. (3.11) and (3.19) into Eq. (3.18), then the kinetic energy of the tip mass can be written as

$$T_t = \frac{1}{2} m_t \left(\dot{X}^2 + \dot{Y}^2 + \alpha^2 + \beta^2 \right)_{s=l} + \frac{1}{2} J_t \left(\dot{\theta} + \dot{v}' \left(1 + \frac{1}{2} v'^2 \right) \Big|_{s=l} \right)^2$$

(3.20)

Substitute Eqs. (3.9) and (3.10) into Eqs. (3.17) and (3.20), then substitute the results and Eq. (3.15) into (3.12) then the total kinetic energy of the system is expressed as

$$\begin{aligned}
T =\ & \frac{1}{2} m_h \dot{X}^2 + \frac{1}{2} m_h \dot{Y}^2 + \frac{1}{4} m_h R^2 \dot{\theta}^2 \\
& + \frac{1}{2} \rho \int_0^l \begin{pmatrix} \dot{X}^2 + \dot{Y}^2 + \\ 2\dot{X}\left[-\left(\dot{\theta}(R + s\cos\eta - v\sin\eta) + \dot{v}\cos\eta \right)\sin\theta - \left(\dot{\theta}(s\sin\eta + v\cos\eta) + \dot{v}\sin\eta \right)\cos\theta \right] \\ + 2\dot{Y}\left[\left(\dot{\theta}(R + s\cos\eta - v\sin\eta) + \dot{v}\cos\eta \right)\cos\theta - \left(\dot{\theta}(s\sin\eta + v\cos\eta) + \dot{v}\sin\eta \right)\sin\theta \right] \\ + \dot{\theta}^2 \left(R^2 + s^2 + v^2 + 2R(s\cos\eta - v\sin\eta) \right) + 2\dot{\theta}\dot{v}(R\cos\eta + s) + \dot{v}^2 \end{pmatrix} ds \\
& + \frac{1}{2} m_t \begin{pmatrix} \dot{X}^2 + \dot{Y}^2 + \\ 2\dot{X}\left[-\left(\dot{\theta}(R + s\cos\eta - v\sin\eta) + \dot{v}\cos\eta \right)\sin\theta - \left(\dot{\theta}(s\sin\eta + v\cos\eta) + \dot{v}\sin\eta \right)\cos\theta \right] \\ + 2\dot{Y}\left[\left(\dot{\theta}(R + s\cos\eta - v\sin\eta) + \dot{v}\cos\eta \right)\cos\theta - \left(\dot{\theta}(s\sin\eta + v\cos\eta) + \dot{v}\sin\eta \right)\sin\theta \right] \\ + \dot{\theta}^2 \left(R^2 + s^2 + v^2 + 2R(s\cos\eta - v\sin\eta) \right) + 2\dot{\theta}\dot{v}(R\cos\eta + s) + \dot{v}^2 \end{pmatrix} \Bigg|_{s=l} \\
& + \frac{1}{2} J_t \left(\dot{\theta} + \dot{v}' \left(1 + \frac{1}{2} v'^2 \right) \Big|_{s=l} \right)^2
\end{aligned}$$

(3.21)

3.2.3. Nonlinear Curvature

The curvature (κ) is given by (Takahashi, 1979).

$$\kappa = \frac{\left(\dfrac{\partial^2 v}{\partial s^2}\right)}{\sqrt{1-\left(\dfrac{\partial v}{\partial s}\right)^2}} \tag{3.22}$$

Expanding in a Taylor expansion, and retaining the non-linear terms up to an order of four, we have

$$\kappa = \frac{\partial^2 v}{\partial s^2}\left(1+\frac{1}{2}\left(\frac{\partial v}{\partial s}\right)^4\right) \tag{3.23}$$

3.2.4. Inextensibility condition

One can relate the axial shortening due to the transverse deflection as follows (Araft, 1999). We now consider the deformation of an element CD of the beam's neutral axis, which is of length *ds* and located at a distance s from the origin O of the (*x*,*y*) system as shown in figure (2).

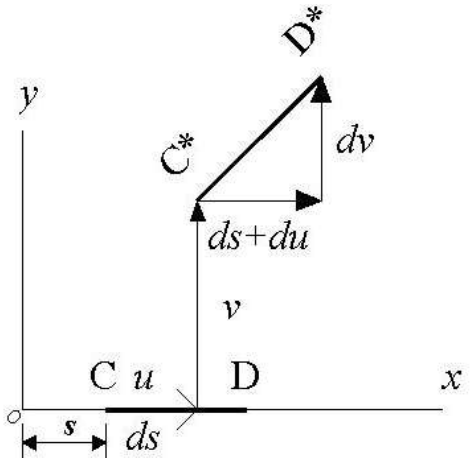

Figure (2) Deformation of a beam element along the neutral axis.

Upon deformation, let CD move to C*D*. we denote the displacement components of C and D by (u, v) and ($u+du$, $v+dv$), respectively. From figure (2) the strain (e) at point C can be calculated as

$$e=\frac{ds^{*}-ds}{ds}=\frac{\sqrt{(ds+du)^{2}+dv^{2}}-ds}{ds}=\sqrt{(1+u')^{2}+v'^{2}}-1 \qquad (3.24)$$

We assume beam neutral axis to be inextensible; that is $e=0$. The Inextensibility constraint equation thus is

$$(1+u')^{2}+v'^{2}=1 \qquad (3.25)$$

Where, primes denote differentiation with respect to dimensional spatial variable s.

Solving for u' and expanding the results in a Taylor expansion, we have

$$u'=-\frac{1}{2}\left(v'^{2}+\frac{1}{4}v'^{4}\right) \qquad (3.26)$$

3.2.5. System Potential Energy

The potential energy of the system (V) consists of three parts potential energy of the hub (V_h), potential energy of the beam (V_b) and potential energy stored in torsional spring (V_t)

$$V=V_h+V_t+V_b \qquad (3.27)$$

The potential energy of the hub is given by (Abed, 2003)

$$V_h=\frac{1}{2}K_xX^{2}+\frac{1}{2}K_yY^{2} \qquad (3.28)$$

Potential energy stored in torsional spring is given by (Abed, 2003)

$$V_t=\frac{1}{2}K_T\left(\frac{\partial v}{\partial s}\right)\Bigg|_{s=0} \qquad (3.29)$$

(K_T) torsional spring constant at beam root and $\left.\frac{\partial v}{\partial s}\right|_{s=0}$ is the slope of deformed beam at its root.

The potential energy of the beam is constituted form the elastic beam strain energy (V_E), and the potential energy of the axial shortening due to transverse deformation and the motion generated inertial force (V_A).

$V_b = V_E + V_A$ (3.30)

The elastic beam strain energy is given by (Hamdan and Al-Bedoor, 2001)

$$V_E = \frac{1}{2}\int_0^l EI\kappa^2 ds \tag{3.31}$$

Substitute (3.22) in (3.31) and retaining the terms up to the fourth order we get

$$V_E = \frac{1}{2}EI\int_0^l \left(v''^2 + v''^2 v'^2\right) ds$$

(3.32)

The centrifugal force on the material point P of the beam (F_p) is given by (Al-Qaisia and Al-Bedoor, 2005)

$$F_p = -\rho\int_s^l \dot{\theta}^2 (R+s) ds \tag{3.33}$$

Upon evaluating the integral given in equation (3.33) the centrifugal force is given by

$$F_p = -\rho\dot{\theta}^2\left[R(l-s) + \frac{1}{2}\left(l^2 - s^2\right)\right] \tag{3.34}$$

The virtual work that results from the axial shortening (u) under the effect of the inertial forces of equation (3.33) can be called the axial shortening potential energy (V_A) and can be written as (Al-Qaisia and Al-Bedoor, 2005)

$$V_A = \int_0^l F_p du \tag{3.35}$$

Substituting the inertial force (3.34) and the axial shortening (3.26) into the integral of equation (3.35) yields

$$V_A = \frac{1}{2}\rho\dot{\theta}^2\int_0^l\left(R(l-s)+\frac{1}{2}\left(l^2-s^2\right)\right)\left(v'^2+\frac{1}{4}v'^4\right)ds \qquad (3.36)$$

Substitute Eqs. (3.32) and (3.36) in Eq. (3.30) then the beam potential energy can be written as

$$V_b = \frac{1}{2}EI\int_0^l\left(v''^2+v''^2v'^2\right)ds+\frac{1}{2}\rho\dot{\theta}^2\int_0^l\left(R(l-s)+\frac{1}{2}\left(l^2-s^2\right)\right)\left(v'^2+\frac{1}{4}v'^4\right)ds \qquad (3.37)$$

Substitute Eqs. (3.28), (3.29) and (3.37) into Eq. (3.27) then the potential energy of the system is given by

$$V = \frac{1}{2}K_xX^2+\frac{1}{2}K_yY^2+\frac{1}{2}K_T\left(\frac{\partial v}{\partial s}\right)\bigg|_{s=0}+\frac{1}{2}EI\int_0^l\left(v''^2+v''^2v'^2\right)ds$$
$$+\frac{1}{2}\rho\dot{\theta}^2\int_0^l\left(R(l-s)+\frac{1}{2}\left(l^2-s^2\right)\right)\left(v'^2+\frac{1}{4}v'^4\right)ds \qquad (3.38)$$

3.2.6 Assumed Mode Method (AMM)

In the analysis of a nonlinear continuous system, which has an infinite number of degrees of freedom, a modal discretization is often employed to obtain a reduced-order model of the system. Here we used the dimensionless AMM in particular which is used in discretizing the beam elastic deformation, *v(s,t)* as follows (Lee, 1994)

$$v(\varsigma,t) = l\phi(\varsigma)q(t) \qquad (3.39)$$

Where $\varsigma = \frac{s}{l}$, *q(t)* is an unknown time modulation of the assumed deflection mode $\phi(\varsigma)$. $\phi(\varsigma)$ is spatial function, for the present study, $\phi(\varsigma)$ is assumed to be that of non-rotating liner beam which can be written in the form

$$\phi_i(\varsigma)=C_1\sin p_i\varsigma+C_2\cos p_i\varsigma+C_3\sinh p_i\varsigma+C_4\cosh p_i\varsigma \qquad (3.40)$$

Where C_1, C_2, C_3, C_4 are arbitrary constants to be determined by using the following four boundary conditions (Karunendiran, 1999).

$$v(0,t)=0$$

$$v''(0)=\frac{K_T l}{EI}v'(0,t)$$

$$v'''(1,t)=-\frac{m_t}{\rho l}p_i^4 v(1,t)$$

$$v''(1,t)=\frac{J_t}{\rho l^3}p_i^4 v'(1,t) \qquad (3.41)$$

Where, primes denote differentiation with respect to non-dimensional spatial variable ς.

Substituting (3.41) into (3.40) we get

$$\phi_i(\varsigma)=-\cos p_i\varsigma+\cosh p_i\varsigma+C_i(\sinh p_i\varsigma-\sin p_i\varsigma)-\frac{2p_i}{a_4}\sin p_i\varsigma \qquad (3.42)$$

Where C_i is given by

$$C_i=\frac{-\left[-\frac{2p_i}{a_4}\cos p_i-\sin p_i+\sinh p_i+a_2p_i\left(\frac{2p_i}{a_4}\sin p_i-\cos p_i+\cosh p_i\right)\right]}{\cos p_i+\cosh p_i+a_2p_i(\sinh p_i-\sin p_i)} \qquad (3.43)$$

And the p_i are the roots of the equation

$$\begin{aligned}&a_4\cos p\cosh p-p\sin p\cosh p+p\cos p\sinh p+a_2a_4p\cos p\sinh p\\&-a_2a_4p\sin p\cosh p-2a_2p^2\sin p\sinh p-a_3a_4p^3\sin p\cosh p-\\&a_3a_4p^3\cos p\sinh p-a_2a_3a_4p^4\cos p\cosh p-2a_3p^4\cos p\cosh p\\&-a_2a_3p^5\cos p\sinh p+a_2a_3p^5\sin p\cosh p+a_2a_3a_4p^4+a_4=0 \qquad (3.44)\end{aligned}$$

3.2.7. The Lagrangian Equations of Motions

For an n degree of freedom system, with n generalized coordinates, equations of motion can be obtained via Lagrange's

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{b}}\right)-\frac{\partial L}{\partial b}=0 \qquad (3.45)$$

Where b is the generalized coordinate vector and is composed of $b=[q, \theta, X, Y]$, and (L) is the Lagrangian, and is defined as the difference between the kinetic and potential energy

$$L=T-V \qquad (3.46)$$

Substitute Eq. (3.39) in Eqs. (3.21) and (3.38) and then substitute the results in (3.46) and using the non-dimensional quantities $s=\frac{\varsigma}{l}, X=\frac{X}{l}$ and $Y=\frac{Y}{l}$ then the Lagrangian can be written as

$$L=\frac{m_b l^2}{2}\begin{pmatrix} \beta_1\dot{\theta}^2+\beta_2\dot{\theta}^2 q+\beta_3\dot{\theta}^2 q^2+\beta_4\dot{\theta}^2 q^4+\beta_5\dot{q}^2+\beta_6\dot{q}^2 q^2 \\ +\beta_7\dot{q}^2 q^4+\beta_8\dot{\theta}\dot{q}+\beta_9\dot{\theta}\dot{q}q^2-\beta^2\beta_{10}q^2-\beta^2\beta_{11}q^4 \\ -\beta^2 a_5 X^2-\beta^2 a_6 Y^2+\beta_{12}\dot{X}^2+\beta_{12}\dot{Y}^2+ \\ \beta_{13}\dot{\theta}\left(\dot{X}\sin\theta-\dot{Y}\cos\theta\right)+\beta_{14}\dot{\theta}\left(\dot{X}\cos\theta+\dot{Y}\sin\theta\right) \\ \beta_{15}\dot{q}\left(\dot{X}\sin\theta-\dot{Y}\cos\theta\right)+\beta_{16}\dot{q}\left(\dot{X}\cos\theta+\dot{Y}\sin\theta\right) \\ +\beta_{15}\dot{\theta}q\left(\dot{X}\cos\theta+\dot{Y}\sin\theta\right)+\beta_{16}\dot{\theta}q\left(-\dot{X}\sin\theta+\dot{Y}\cos\theta\right) \end{pmatrix} \qquad (3.47)$$

Where the bar over X and Y has been dropped for ease of notation.

Where

$$a_1=\frac{m_h}{m_b} \qquad a_2=\frac{m_t}{m_b} \qquad a_3=\frac{J_t}{m_b l^2} \qquad a_4=\frac{K_T l}{EI}$$

$$a_5=\frac{K_x l}{EI} \qquad a_6=\frac{K_y l}{EI} \qquad \beta^2=\frac{EI}{m_b l^3}$$

$$\beta_1=\left(\left(\frac{R}{l}\right)^2+\frac{R}{l}\cos\eta+\frac{1}{3}\right)+a_2\left(\left(\frac{R}{l}\right)^2+2\frac{R}{l}\cos\eta+1\right)+\frac{1}{2}a_1\left(\frac{R}{l}\right)^2+a_3$$

$$\beta_2=-2\frac{R}{l}\sin\eta\left(\int_0^1\phi\, d\varsigma+a_2\phi|_{\varsigma=1}\right)$$

$$\beta_3 = \int_0^1 \phi^2 d\varsigma + a_2\phi^2\Big|_{\varsigma=1} + \int_0^1 \left(\frac{R}{l}(1-\varsigma) + \frac{1}{2}(1-\varsigma^2)\right)\phi'^2 d\varsigma$$

$$\beta_4 = \frac{1}{4}\int_0^1 \left(\frac{R}{l}(1-\varsigma) + \frac{1}{2}(1-\varsigma^2)\right)\phi'^4 d\varsigma$$

$$\beta_5 = \int_0^1 \phi^2 d\varsigma + a_2\phi^2\Big|_{\varsigma=1} + a_3\phi'^2\Big|_{\varsigma=1}$$

$$\beta_6 = a_3\phi'^4\Big|_{\varsigma=1}$$

$$\beta_7 = \frac{1}{4}a_3\phi'^6\Big|_{\varsigma=1} \quad \beta_8 = 2\int_0^1 \phi\left(\frac{R}{l}\cos\eta + \varsigma\right)d\varsigma + 2a_2\left(\frac{R}{l}\cos\eta + \varsigma\right)\phi\Big|_{\varsigma=1} + 2a_3\phi'\Big|_{\varsigma=1}$$

$$\beta_9 = a_3\phi'^3\Big|_{\varsigma=1}$$

$$\beta_{10} = \int_0^1 \phi''^2 d\varsigma + a_4\phi'\Big|_{\varsigma=0}$$

$$\beta_{11} = \int_0^1 \phi'^2\phi''^2 d\varsigma$$

$$\beta_{12} = a_1 + a_2 + 1$$

$$\beta_{13} = -\left(2\frac{R}{l} + \cos\eta\right) - 2a_2\left(\frac{R}{l} + \cos\eta\right)$$

$$\beta_{14} = -(\sin\eta + 2a_2\sin\eta)$$

$$\beta_{15} = -2\cos\eta\left(\int_0^1 \phi d\varsigma + a_2\phi\Big|_{\varsigma=1}\right)$$

$$\beta_{16} = -2\sin\eta\left(\int_0^1 \phi d\varsigma + a_2\phi\Big|_{\varsigma=1}\right)$$

Substitute Eq. (3.47) in Eq.(3.45) we obtain

$$
\begin{aligned}
&2\ddot{\theta}\left(\beta_1+\beta_2 q+\beta_3 q^2 \beta_4 q^4\right)+\ddot{q}\left(\beta_8+\beta_9 q^2\right)+8\beta_4 q^3 \dot{q}\dot{\theta}+2\beta_9 q\dot{q}^2 \\
&+\ddot{X}\left(\beta_{13}\sin\theta+\beta_{14}\cos\theta+q\left(\beta_{15}\cos\theta-\beta_{16}\sin\theta\right)\right) \\
&+\ddot{Y}\left(\beta_{14}\sin\theta-\beta_{13}\cos\theta+q\left(\beta_{15}\sin\theta+\beta_{16}\cos\theta\right)\right)=0
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
&2\ddot{q}\left(\beta_5+\beta_6 q^2+\beta_7 q^4\right)+\ddot{\theta}\left(\beta_8+\beta_9 q^2\right)+\ddot{X}\left(\beta_{15}\sin\theta+\beta_{16}\cos\theta\right) \\
&+\ddot{Y}\left(\beta_{16}\sin\theta-\beta_{15}\cos\theta\right)-\dot{\theta}^2\left(\beta_2+2\beta_3 q+4\beta_4 q^3\right)+2\beta_6 q\dot{q}^2 \\
&+4\beta_7 q^3\dot{q}^2+2\beta^2\beta_{10}q+4\beta^2\beta_{11}q^3=0
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
&2\beta_{12}\ddot{X}+\ddot{\theta}\left(\beta_{13}\sin\theta+\beta_{14}\cos\theta+q(\beta_{15}\cos\theta-\beta_{16}\sin\theta)\right) \\
&\ddot{q}\left(\beta_{15}\sin\theta+\beta_{16}\cos\theta\right)+2\dot{q}\dot{\theta}\left(\beta_{15}\cos\theta-\beta_{16}\sin\theta\right) \\
&+\dot{\theta}^2\left(\beta_{13}\cos\theta-\beta_{14}\sin\theta-q\left(\beta_{15}\sin\theta+\beta_{16}\cos\theta\right)\right)=0
\end{aligned}
\tag{3.50}
$$

$$
\begin{aligned}
&2\beta_{12}\ddot{Y}+\ddot{\theta}\left(\beta_{14}\sin\theta-\beta_{13}\cos\theta+q\left(\beta_{15}\sin\theta+\beta_{16}\cos\theta\right)\right) \\
&+\ddot{q}\left(\beta_{16}\sin\theta-\beta_{15}\cos\theta\right)+2\dot{q}\dot{\theta}\left(\beta_{15}\sin\theta+\beta_{16}\cos\theta\right) \\
&+\dot{\theta}^2\left(\beta_{13}\sin\theta+\beta_{14}\cos\theta+q\left(\beta_{15}\cos\theta-\beta_{16}\sin\theta\right)\right)+2a_6\beta^2 Y=0
\end{aligned}
\tag{3.51}
$$

The above four coupled and non-linear equations are difficult to solve as they stand. And thus a simplification is needed to enable one to obtain a reasonable understanding of the system behavior. Assuming that the hub rotates at constant angular velocity then $\ddot{\theta}=0$, assuming that the linear springs (elastic foundation) in X and Y directions is prescribed by a harmonic functions

$$X=\Lambda_o \sin\omega_e^* t \tag{3.52}$$

$$Y=\Lambda_o \cos\omega_e^* t \tag{3.53}$$

Respectively.

Where Λ_o and $\omega^*{}_e$ are the amplitude and frequency of the harmonic functions.

3.2.8 Dimensionless Equations of Motion

To lend generality to the numerical results equations are to be transformed into dimensionless equations. For this transformation, dimensionless time parameter is defined as follows:

$$t^{*}=\omega_{n}t \qquad (3.54)$$

Where $\omega_{n}=\left(\frac{\beta^{2}\beta_{10}}{\beta_{5}}\right)^{\frac{1}{2}}$

Substitute Eqs. (3.52) and (3.53) into Eq.(3.49) and transforms the derivatives form t to t^{*} then Eq. (3.48) can be written as

$$\ddot{u}+u-\varepsilon_{1}\Omega^{2}u-\varepsilon_{2}\Omega^{2}u^{3}+\varepsilon_{3}\left(u^{2}\ddot{u}+u\dot{u}^{2}\right)+\varepsilon_{4}u^{3}+\varepsilon_{5}\left(u^{4}\ddot{u}+2\dot{u}^{2}u^{3}\right)-\varepsilon_{o}\Omega^{2}=\Lambda\omega_{e}^{2}\cos\left(\left(\omega_{e}+\Omega\right)t^{*}+\varphi\right)$$

(3.55)

Where

$$\varepsilon_{o}=\frac{p_{i}\beta_{2}}{2\beta_{5}} \qquad \varepsilon_{1}=\frac{\beta_{3}}{\beta_{5}} \qquad \varepsilon 2=\frac{2\beta_{4}}{\beta_{5}p_{i}^{2}}$$

$$\varepsilon_{3}=\frac{\beta_{6}}{\beta_{5}p_{i}^{2}} \qquad \varepsilon_{4}=\frac{2\beta_{11}}{\beta_{10}p_{i}^{2}} \qquad \varepsilon_{5}=\frac{\beta_{7}}{\beta_{5}p_{i}^{4}}$$

$$\Lambda=\frac{p}{2\beta_{5}}\sqrt{\beta_{15}^{2}+\beta_{16}^{2}} \qquad \varphi=\tan^{-1}\left(\frac{\beta_{16}}{\beta_{15}}\right)$$

Here dots derivative with respect to non-dimensional time t^{*}. $u=p_{i}q$ is a dimensionless tip displacement, $\Omega=\frac{\dot{\theta}}{\omega_{ni}}$ is a dimensionless hub speed ratio, and $\omega_{e}=\frac{\omega_{e}^{*}}{\omega_{n}}$ is a dimensionless excitation frequency.

3.3. Methods of Solution

In this section, the relation between nonlinear natural frequencies and rotating speed for rotating beam is obtained using the method of multiple scales, and the method of harmonic balance. Hub radius beam length ratio, tip mass, rotary inertia, attachment angle, and flexible root are investigated.

3.3.1. The Method of Multiple Scales

The origins of the method of multiple scales go back to Krylov and Bogoliubov in 1932 (Cartmell. *et al*., 2003) The general principle behind the method is that dependent variable(s) is (are) uniformly expanded in terms of two or more independent variables, nominally referred to as scales. This obviously requires that the time derivatives of the dependent variable(s) are similarly expressed, with the general consequence that uniformity is relatively well preserved. (Cartmell. *et al*., 2003)

The multiple independent variables are generated with respect to real (clock) time t such that

$$T_n = \varepsilon^n t \quad \text{for } n = 0,1,2,\ldots \qquad (3.56)$$

Clearly, then, the time derivatives will be expansions, in their own right, in terms of partial derivatives, each with respect to the T_n as follows:

$$\frac{d}{dt} = \frac{\partial}{\partial T_o} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial^2}{\partial T_2} + \ldots = D_o + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_o^2} + 2\varepsilon \frac{\partial^2}{\partial T_o \partial T_1} = D_o^2 + 2\varepsilon D_o D_1 + \varepsilon^2 \left(D_1^2 + 2 D_o D_2 \right) + \ldots$$

(3.57)

where

$$D_n = \frac{\partial}{\partial T_n} \qquad n = 0,1,2,\ldots$$

The dependent variables are typically expressed by the following

$$u(t,\varepsilon) = \sum_{j=0}^{m-1} \varepsilon^j u_j \left(T_{o,} T_1, \ldots T_m \right) + O(\varepsilon T_m) \qquad (3.58)$$

Substituting of (3.57) and (3.58) into the governing equations of motion results in a set of perturbation equations, hierarchically order to ε , which can then be solved successively. Solution can proceed form this point, notwithstanding the removal of secular terms (terms whose structure results in nonuniformity of the power series solution). Uniformity is assured if $\varepsilon^j u_j$ =. Secular terms are routinely identified, by means of recognizable resonance conditions, removed, and then equated to zero. This process in invariably used to determine the amplitudes A of the zeros order perturbation solution (sometimes called the generating solution) whereby, for j=0, and expressed in complex form (for later algebraic expediency),

$$u_o = Ae^{i\omega T_o} + \bar{A}e^{-i\omega T_o} \tag{3.59}$$

Noting that ω is an appropriate natural frequency of un-damped vibration and that the over-bar represents complex conjugacy. Once this stage has been reached, solutions can be obtained, progressively, to higher orders or perturbation for each different resonance condition. Finally, the perturbation solutions for the resonance condition of interest are recombined according to Eq.(3.58) to give an approximate, but frequently very accurate, solution to the dependent variables in the time domain. (Cartmell. *et al*., 2003)

3.3.1.1. First Order Expansion

In this subsection, first order approximation of Eq. (3.55) is obtained for 0 attachment angle and without elastic foundation. In order to apply the method of multiple scales described above, it is necessary to reorder various terms in Eq. (3.55), for example, since the concern is with primary resonance, the non-linear terms, are multiplied by a small gauge parameter, ε, so that they appear at the same order in the perturbation scheme (Al-Qaisia and Hamdan, 1999). Accordingly Eq. (3.55) becomes;

$$\ddot{u}+u-\varepsilon_1\Omega^2u-\varepsilon\varepsilon_2\Omega^2u^3+\varepsilon\varepsilon_3\left(u^2\ddot{u}+u\dot{u}^2\right)+\varepsilon\varepsilon_4u^3+\varepsilon\varepsilon_5\left(u^4\ddot{u}+2\dot{u}^2u^3\right)=0 \qquad (3.60)$$

the solution of Eq. (3.60) can represented by put m=2 in Eq. (3.58)

$$u(t;\varepsilon)=u_o(T_o,T_1,..)+\varepsilon u_1(T_o,T_1,..) \qquad (3.61)$$

Substituting Eq. (3.61) into Eq. (3.60), and equating like powers of ε to zero, one obtains

$$\varepsilon^0: \quad D_o^2u_o+u_o\left(1-\varepsilon_1\Omega^2\right) \quad =0 \qquad (3.62)$$

$$\varepsilon^1: \quad D_o^2u_1+u_1\left(1-\varepsilon_1\Omega^2\right) \quad =-2D_oD_1u_o+\varepsilon_2\Omega^2u_o^3-\varepsilon_3\left(u_o\left(D_ou_o\right)^2+u_o^2D_o^2u_o\right)$$
$$-\varepsilon_4u_o^3-\varepsilon_5\left(u_o^4D_o^2u_o+2u_o^3\left(D_ou_o\right)^2\right)$$

(3.63)

Let $m^2=1-\varepsilon_1\Omega^2$. Then the solution of Eq.(3.62) can be written as

$$u_o(T_o,T_1)=A(T_1)e^{imT_o}+\bar{A}(T_1)e^{-imT_o} \qquad (3.64)$$

Where A is an unknown complex function and $\bar{A}$ is the complex conjugate of A. substituting Eq. (3.64) into Eq. (3.63) leads to

$$D_o^2u_1+u_1\left(1-\varepsilon_1\Omega^2\right)=-2D_1Aime^{imT_o}+\varepsilon_2\Omega^2\left(A^3e^{3imT_o}+3A^2\bar{A}e^{imT_o}\right) \qquad (3.65)$$
$$-\varepsilon_3\left(-2A^3m^2e^{3imT_o}-2A^2\bar{A}m^2e^{imT_o}\right)-\varepsilon_4\left(A^3e^{3imT_o}+3A^2\bar{A}e^{imT_o}\right)$$
$$-\varepsilon_5\left(-3m^2A^5e^{5imT_o}-7m^2A^4\bar{A}e^{3imT_o}-6m^2A^3\bar{A}^2e^{imT_o}\right)+cc$$

Where the cc denotes the complex conjugate of the preceding terms.

Eliminating the terms in Eq. (3.65) that produce secular terms in u_1 yields

$$-2D_1Aim+3\varepsilon_2\Omega^2A^2\bar{A}+2\varepsilon_3A^2\bar{A}m^2-3\varepsilon_4A^2\bar{A}+6\varepsilon_5m^2A^3\bar{A}=0$$

(3.66)

Write A in polar form

$$A=\frac{1}{2}ae^{i\beta} \qquad (3.67)$$

Where a and β are real function of T_1. Substituting (3.67) into (3.66) and separating the result into real and imaginary parts, we obtain

$$a' = 0$$

$$\beta' = -\frac{3}{8m}\varepsilon_2\Omega^2 a^2 - \frac{1}{4}\varepsilon_3 m a^2 + \frac{3}{8m}\varepsilon_4 a^2 - \frac{3}{16} m a^4$$

Where the prime denotes the derivative with respect to T_1. It follows that a is a constant and hence that

$$\beta = \left(-\frac{3}{8m}\varepsilon_2\Omega^2 a^2 - \frac{1}{4}\varepsilon_3 m a^2 + \frac{3}{8m}\varepsilon_4 a^2 - \frac{3}{16} m a^4\right)T_1 + \beta_o \qquad (3.68)$$

Where β_o is a constant. Substituting Eq. (3.67) in Eq. (3.66), we find that

$$A = \frac{1}{2}a\exp\left(i\left(-\frac{3}{8m}\varepsilon_2\Omega^2 a^2 - \frac{1}{4}\varepsilon_3 m a^2 + \frac{3}{8m}\varepsilon_4 a^2 - \frac{3}{16} m a^4\right)\varepsilon t + i\beta_o\right)$$

(3.69)

Substituting for u_o from Eq. (3.64) into Eq. (3.61) and using the Eq. (3.69), we obtain

$$u = a\cos(\omega t + \beta_o) \qquad (3.70)$$

where

$$\omega = \left(-\frac{3}{8m}\varepsilon_2\Omega^2 a^2 - \frac{1}{4}\varepsilon_3 m a^2 + \frac{3}{8m}\varepsilon_4 a^2 - \frac{3}{16} m a^4\right)\varepsilon + m \qquad (3.71)$$

is the non-dimensional non linear natural frequency.

3.3.2 The Method of Harmonic Balance

The harmonic balance method facilitates the investigation of a periodic response from any system. The idea is to express the periodic solution of the equation of any system in the form.

$$u = \sum_{m=0}^{M} A_m \cos(m\omega t) \qquad (3.72)$$

and then substituting in the equation of the system and equating the coefficient of each of the lowest M+1 harmonics to zero, we obtain a system of M+1 algebraic equations relating ω and the A_m. Usually these equations are solved for A_o, A_1, A_2,…,A_m and ω in term of A_1. The accuracy of the resulting periodic solution depends on the value of A_1 and the number of harmonics in the assumed solution (Nayfeh, 1979)

3.3.2.1 Single Term Harmonic Balance (SHM)

In this subsection, non-linear natural frequency of Eq. (3.55) is obtained for 0 attachment angle and without elastic foundation then Eq.(3.55) becomes.

$$\ddot{u}+u-\varepsilon_1\Omega^2u-\varepsilon_2\Omega^2u^3+\varepsilon_3\left(u^2\ddot{u}+u\dot{u}^2\right)+\varepsilon_4u^3+\varepsilon_5\left(u^4\ddot{u}+2\dot{u}^2u^3\right)=0 \quad (3.73)$$

One-term expansion can be expressed as

$$u=A_1\cos(\omega t) \quad (3.74)$$

Where A_1 is constant amplitude and ω is the dimensionless natural frequency.

Substituting (3.74) and its time derivatives into Eq. (3.73), collecting the coefficient of cos(ωt) and equating them to zero, one obtains the relation

$$\omega^2=\frac{-1+\varepsilon_1\Omega^2+\frac{3}{4}\varepsilon_2A_1^2\left(\Omega^2-\varepsilon_4\right)}{-1-\frac{1}{2}\varepsilon_3A_1^2-\frac{3}{8}\varepsilon_5A_1^4} \quad (3.75)$$

3.3.2.2 Two Term Harmonic Balance (THM)

In this subsection, non-linear natural frequencies are obtained for any attachment angle and without elastic foundation then Eq.(3.55) becomes.

$$\ddot{u}+u-\varepsilon_1\Omega^2u-\varepsilon_2\Omega^2u^3+\varepsilon_3\left(u^2\ddot{u}+u\dot{u}^2\right)+\varepsilon_4u^3+\varepsilon_5\left(u^4\ddot{u}+2\dot{u}^2u^3\right)-\varepsilon_o\Omega^2=0 \quad (3.76)$$

In order to study the effect of attachment angle, we need to include other terms besides the first harmonic following (Al-Qaisia and Hamdan, 2001) and putting

$$u=A_o+A_1\cos(\omega t) \quad (3.77)$$

Where A_o is a constant bias.

Substituting Eq. (3.77) and its time derivatives into Eq. (3.76), collecting the constant terms and the coefficients of cos(ωt) and equating them to zero, one obtains the following set of non-linear algebraic equations for the unknowns A_o, A_1, and ω.

$$\omega^2 = \frac{-1+\varepsilon_1\Omega^2+\varepsilon_2\Omega^2\left(3A_o^2+\frac{3}{4}A_1^2\right)-\varepsilon_4\left(3A_o^2+\frac{3}{4}A_1^2\right)}{-1-\varepsilon_3\left(\frac{1}{2}A_1^2+A_o^2\right)-\varepsilon_5\left(A_o^4+3A_o^2A_1^2+\frac{3}{8}A_1^4\right)} \tag{3.78}$$

$$A_o = \frac{\varepsilon_o\Omega^2+A_o^3\left(\varepsilon_2\Omega^2-\varepsilon_4+\varepsilon_5A_1^2\omega^2\right)}{1-\varepsilon_1\Omega^2-\frac{3}{2}\varepsilon_2\Omega^2A_1^2-\frac{1}{2}\varepsilon_3A_1^2\omega^2+\frac{3}{2}\varepsilon_4A_1^2-\frac{3}{4}\varepsilon_5A_1^4\omega^2} \tag{3.79}$$

Results

In this section the effect of root radius, tip mass, on the variation of the dimensionless non-linear natural frequency with the dimensionless rotational speed are presented in figures using the method of multiple scale and harmonic balance

4.1 Numerical Results

In table (2) some of the results are compared with available results obtained by (Eric,2002).The parameters used in computing non-linear natural frequencies are those given in (Eric,2002). l=9m, ρ=10 kg/m , EI=3.99×10^5 $N.m^2$ and R=0.5m.

Table (2)The first three non linear natural frequencies (Hz) of a rotating uniform beam for different rotating speed

	0 rad/sec			30 rad/sec		
Method	1^{st} mode	2^{nd} mode	3^{rd} mode	1^{st} mode	2^{nd} mode	3^{rd} mode
MMS	8.671	54.339	152.15	18.104	91.190	198.925
SHB	8.678	54.402	152.309	18.153	91.290	199.109
(Eric,2002).	8.672	54.35	152.2	34.03	95.84	200.5

4.2 Effect of the Root Radius

Figs. (3-5) shows the influence of the dimensionless rotating speed and hub radius ratio (R/l) on the first three non-linear natural frequencies of a uniformly cantilever beam with parameters (a_2=0, a_3=0,a_4=∞ η=0 and (R/l)=0.25). Using the (MMS) Eq.(3.71) and (SHB) Eq. (3.75).

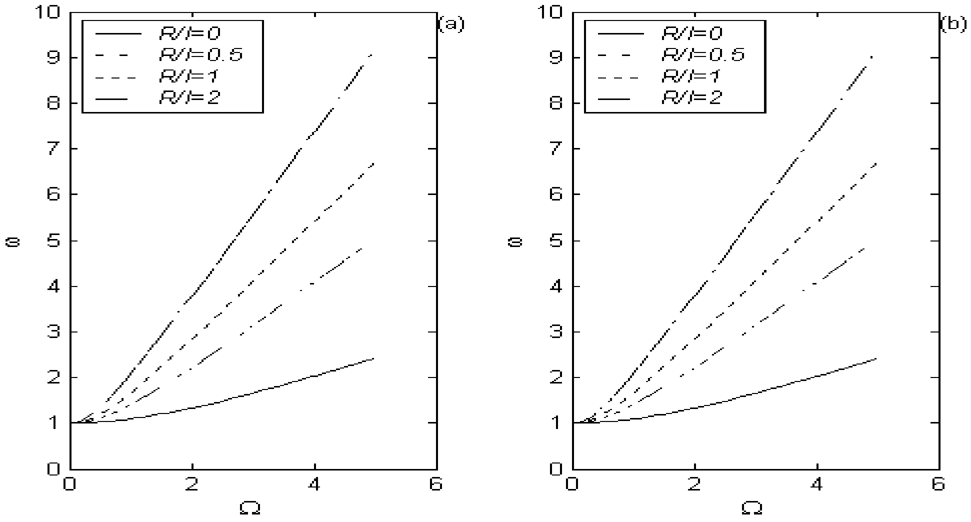

Figure (3): Variation of the first natural frequencies with angular velocity for different values of (*R/l*), using: (a) SHB (b) MMS.

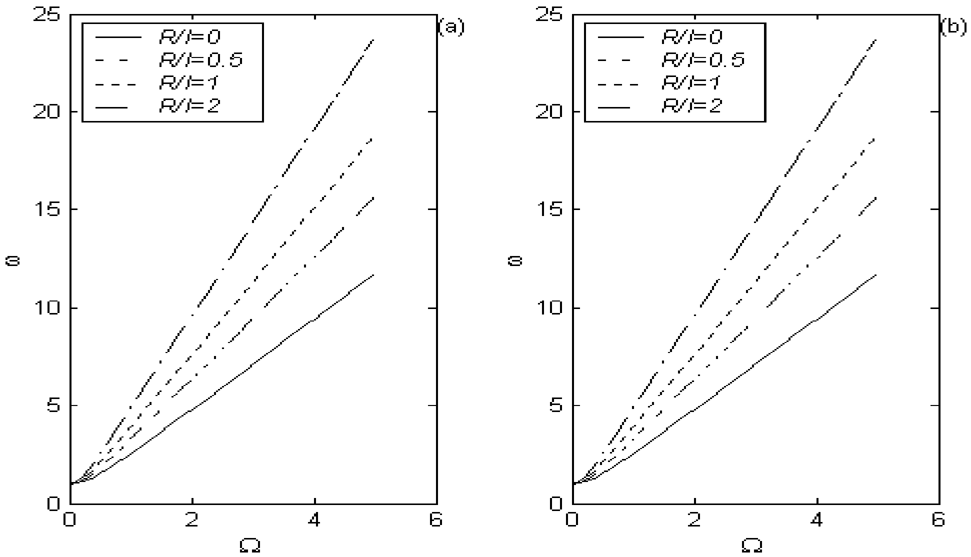

Figure (4): Variation of the second natural frequencies with angular velocity for different values of (R/l), using: (a) SHB (b) MMS.

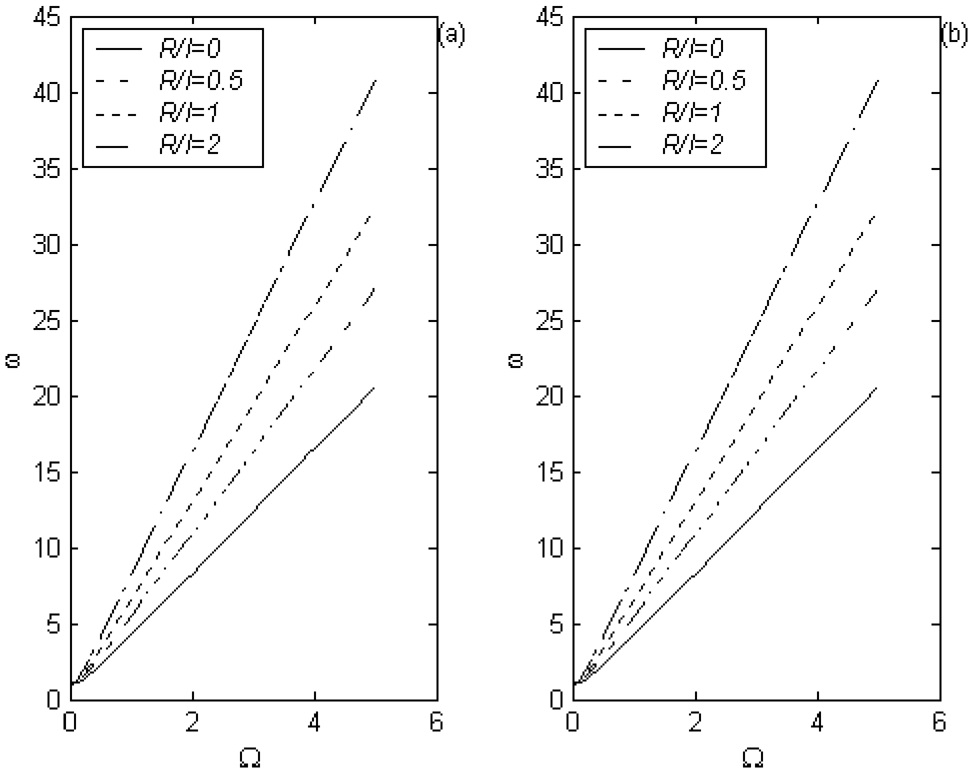

Figure (5): Variation of the third natural frequencies with angular velocity for different values of (R/l), using: (a) SHB (b) MMS. *4.3 Effect of Root Spring Constant*

Figs. (6-8) show the influence of the torsional root spring constant (a_4) on the first three non-linear natural frequencies of a rotating beam with parameters (a_2=0, a_3=0, η=0 and (R/l)=0.25) using the (MMS) Eq.(3.71) and (SHB) Eq. (3.75).

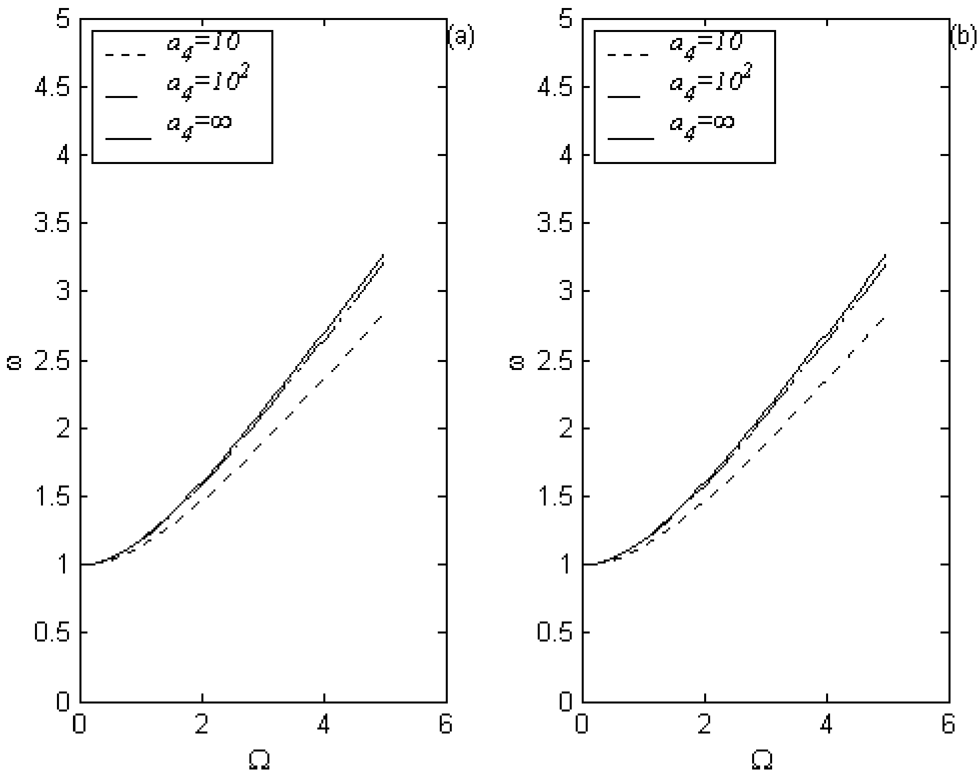

Figure (6):The influence of the angular velocity (Ω) and rotational root spring constant (a_4) on the second non-linear natural frequency. (a) SHB (b) MMS.

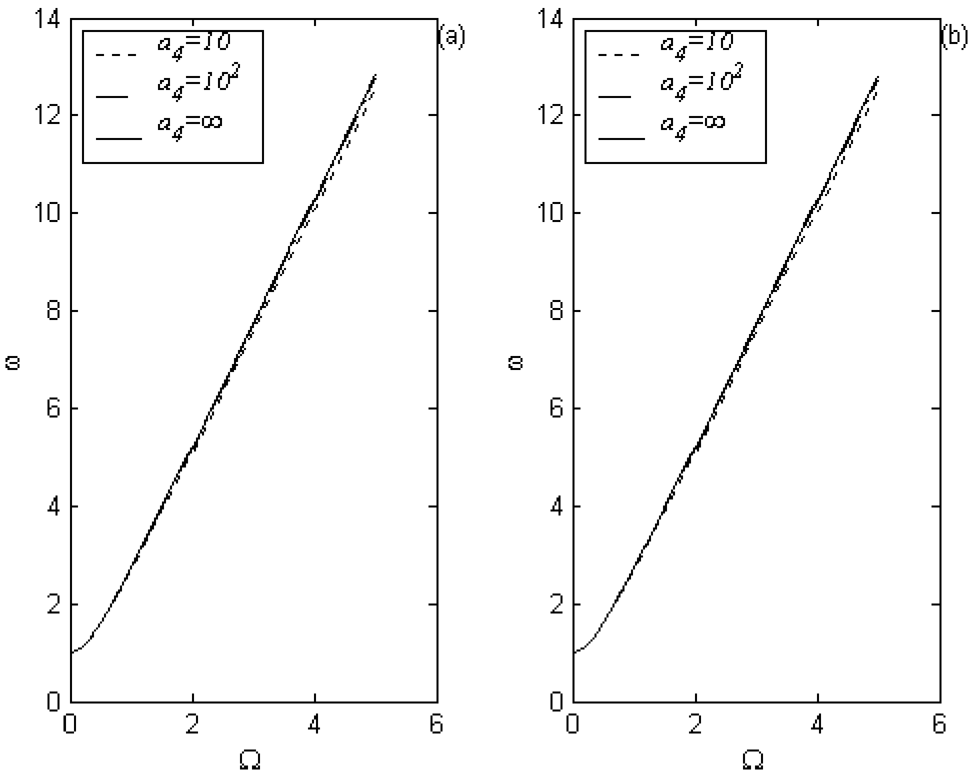

Figure (7): The influence of the angular velocity (Ω) and rotational root spring constant (a_4) on the second non-linear natural frequency. (a) SHB (b) MMS.

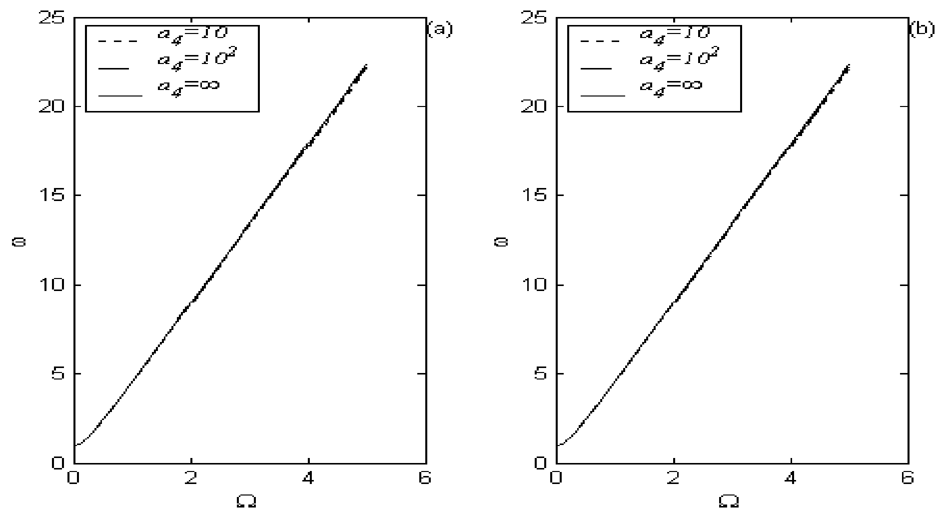

Figure: (8): The influence of the angular velocity (Ω) and rotational root spring constant (a_4) on the third non-linear natural frequency. (a) SHB (b) MMS

4.4 Effect of the Attachment Angle

Figs. (9-11) show the influence of the attachment angle (η) on the first three non-linear natural frequencies of a uniformly rotating cantilever beam with parameters (a_2=0, a_3=0, a_4=∞ and (R/l)=0.25). Using the Eqs. (3.78 and 3.79) which is solved numerically by direct iteration techniques with 10^{-6} accuracy

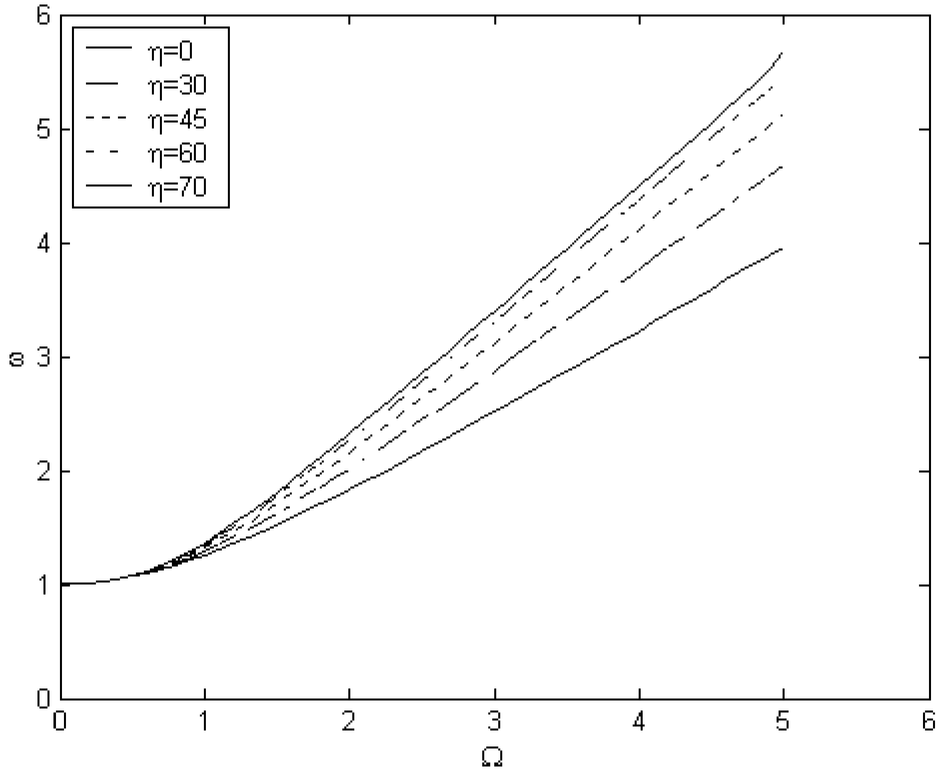

Figure (9):The influence of the angular velocity (Ω) and setting angle (η) on the first t non-linear natural frequency.

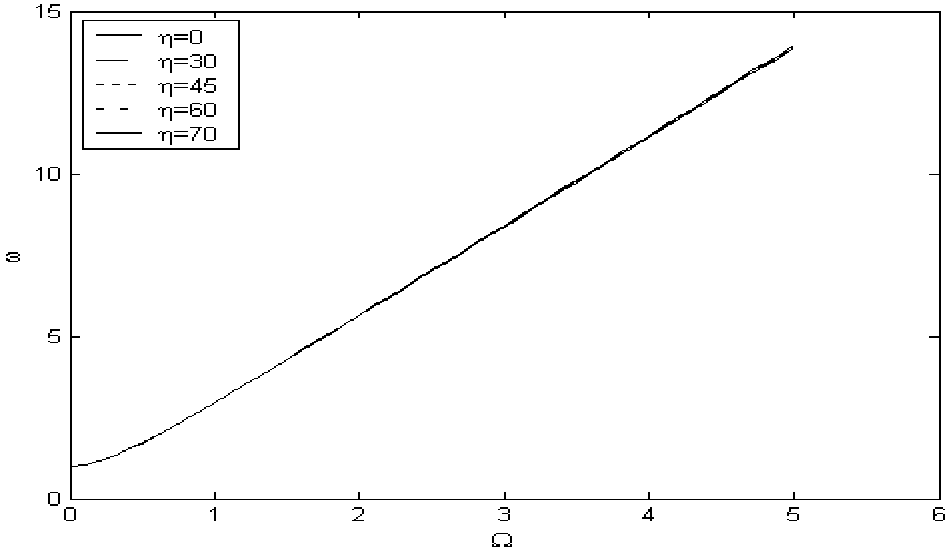

Figure (10):The influence of the angular velocity (Ω) and setting angle (η) on the second non-linear natural frequency.

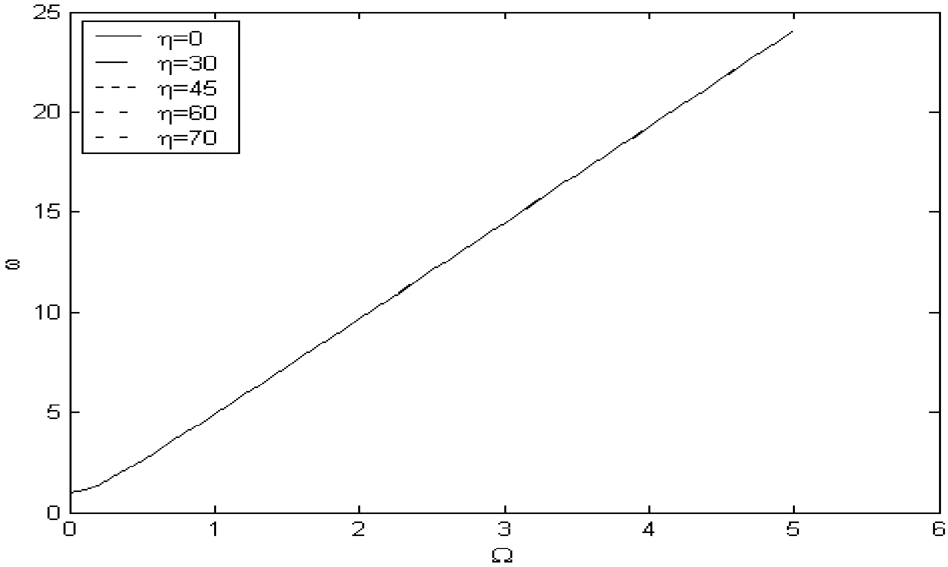

Figure (11):The influence of the angular velocity (Ω) and setting angle (η) on the third non-linear natural frequency.

4.5 Effect Tip Mass

The influence of dimensionless rotational speed on the non-linear natural frequencies of a uniformly cantilever beam with parameters (a_3=0, a_4=∞,

η=0 and (R/l)=0.5). with different values of tip mass constant is shown in Figs.(12-14) using the (MMS) Eq.(3.71) and (SHB) Eq. (3.75).

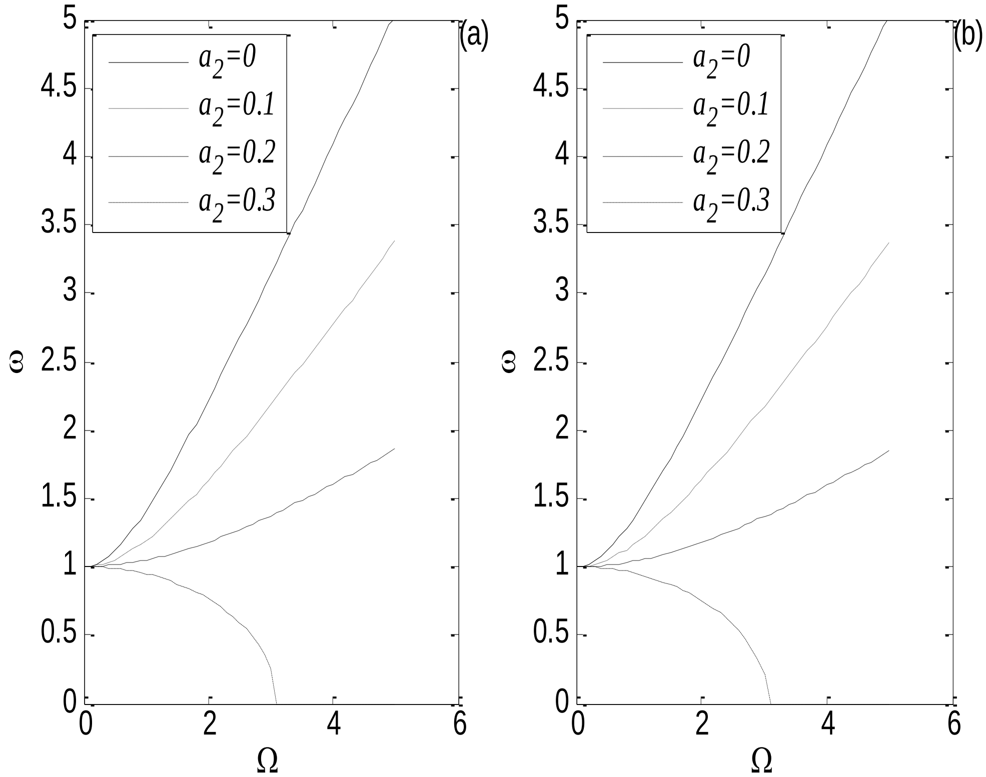

Figure (12): The influence of the angular velocity (Ω) and tip mass constant (a_2) on the first non-linear natural frequency. (a) SHB (b) MMS.

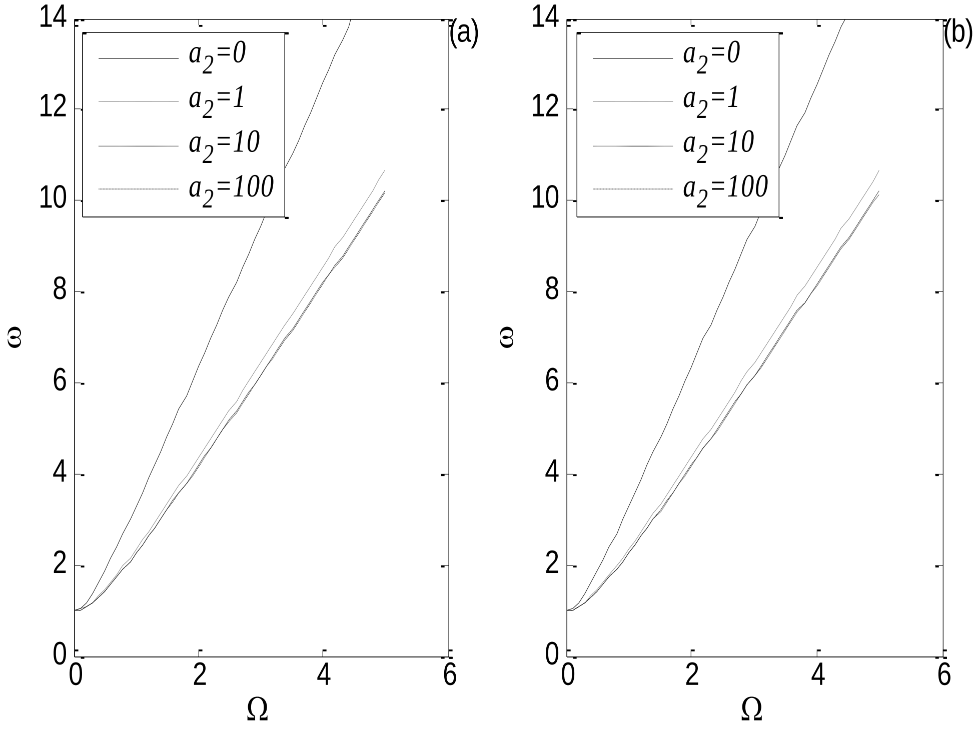

Figure (13): The influence of the angular velocity (Ω) and tip mass constant (a_2) on the second non-linear natural frequency. (a) SHB (b) MMS.

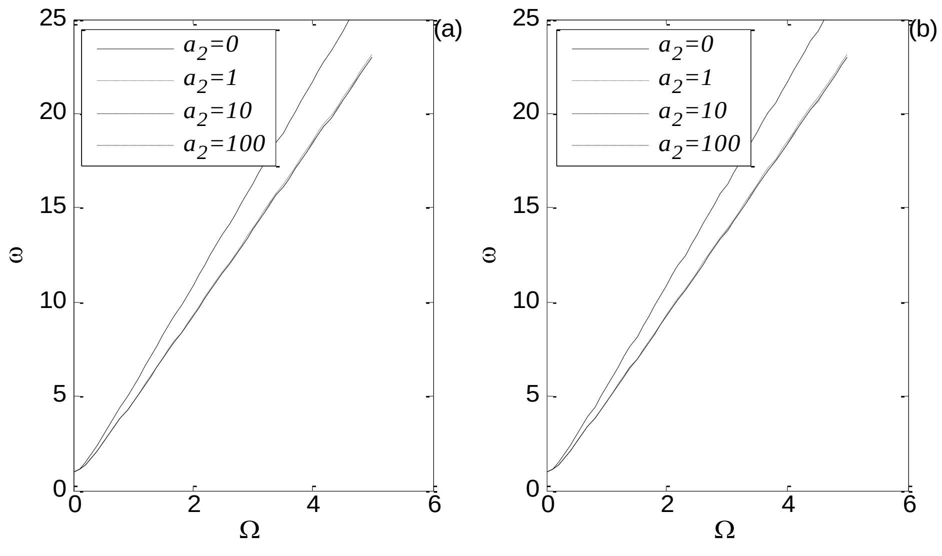

Figure (14): The influence of the angular velocity (Ω) and tip mass constant (a_2) on the third non-linear natural frequency. (a) SHB (b) MMS.

4.6 Effect of Rotary Inertia

The influence of dimensionless rotational speed on the non-linear natural frequencies of a uniformly cantilever beam with parameters (a_2=0.1, $a_4=\infty$, η=0 and (R/l)=0.25) .with tip mass constant having different inertia constans are shown in Figs.(15-17) using the (MMS) Eq.(3.71) and (SHB) Eq. (3.75).

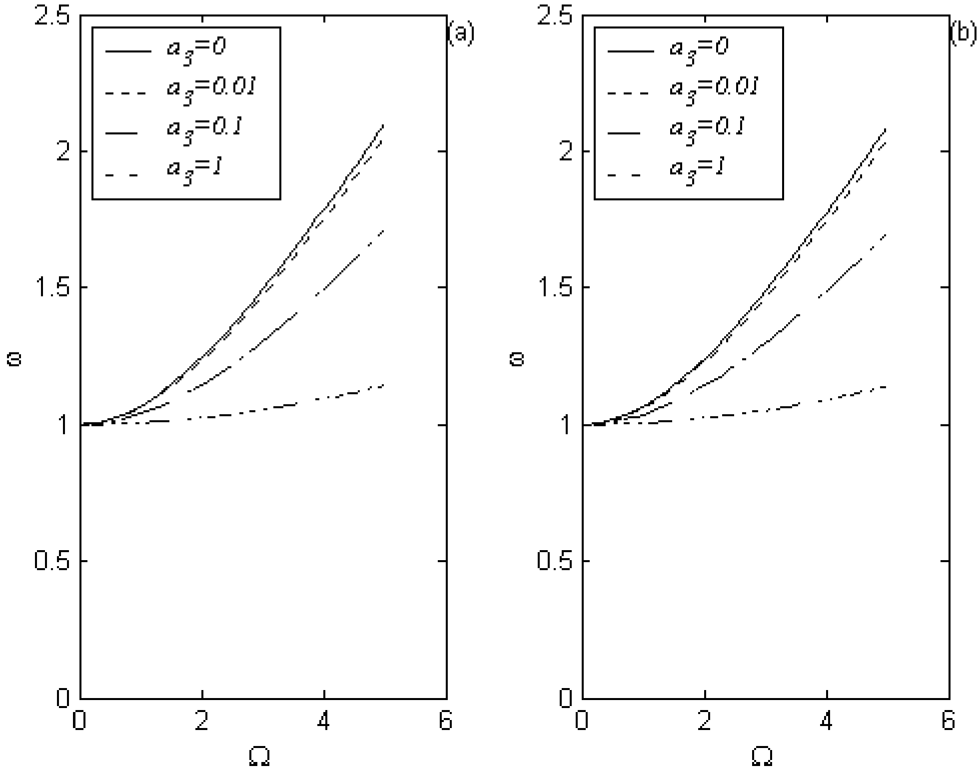

Figure (15): The influence of the angular velocity (Ω) and rotary inertia constants (a_3) on the first non-linear natural frequency. (a) SHB (b) MMS.

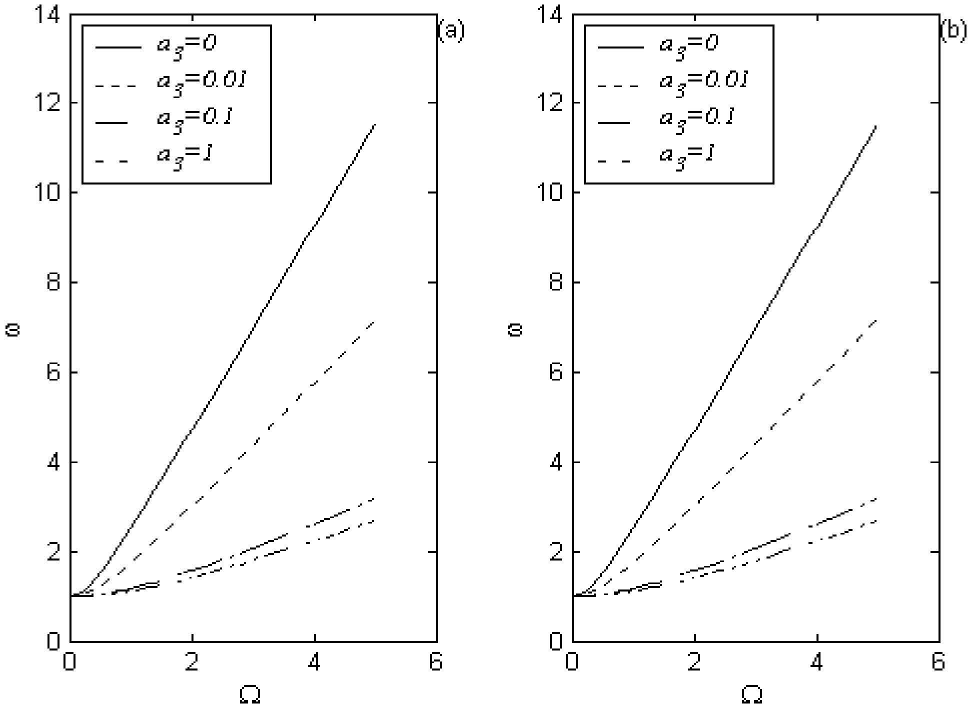

Figure (16): The influence of the angular velocity (Ω) and rotary inertia constants (a_3) on the second non-linear natural frequency. (a) SHB (b) MMS.

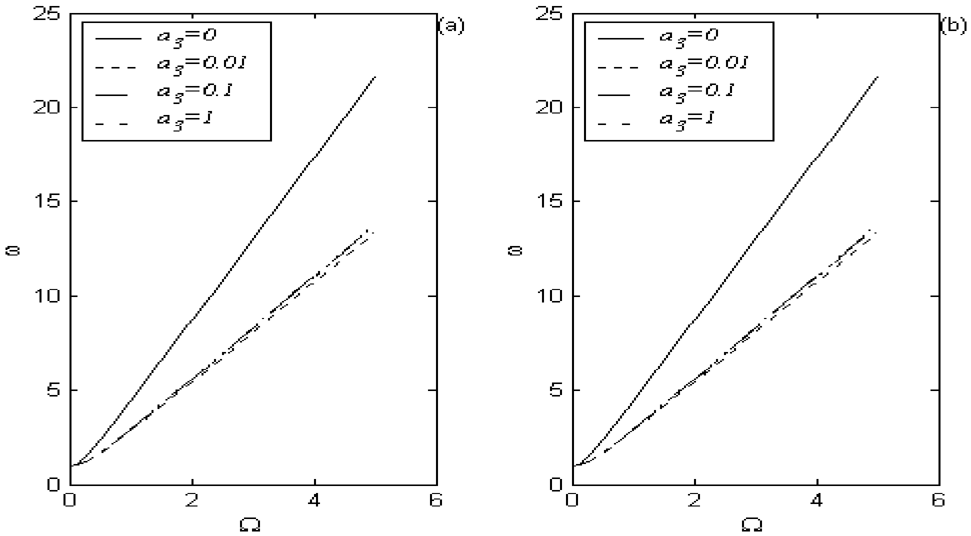

Figure (17): The influence of the angular velocity (Ω) and rotary inertia constants (a_3) on the third non-linear natural frequency. (a) SHB (b) MMS.

Discussion

5.1 Numerical Results

As can be seen form table (1) that (MMS) and (HB) give approximately the same results. It can also be noted that there is a difference in values of non-linear natural frequencies between the present study and (Eric, 2002). Table (2) summarize the percentage of differences between them.

Table (2) Percentage of differences (%) between (Eric, 2002).MMS, and SHB

	0 rad/sec			30 rad/sec		
Method	1st mode	2nd mode	3rd mode	1st mode	2nd mode	3rd mode
MMS	0.01153	0.0202	0.0329	46.79	4.85	0.78
SHB	0.06918	0.0957	0.0716	46.65	4.74	0.69

These differences maybe due to the nonlinear terms including in modeling our model. There is a good agreement in non-linear natural frequencies for non-rotating beam. The percentages of difference become smaller for higher modes.

5.2 Effect of the Root Radius

Figs. (3-5) Show the influence of the dimensionless rotating speed and hub radius ratio on non-linear natural frequencies. It can be noted that the dimensionless non-linear natural frequencies (ω) increase as the angular speed ratio (Ω), and the increasing rate become larger as the hub radius ratio (R/l) become larger. This can be explained by the fact that the centrifugal force increases as the angular speed and the hub radius increase.

5.3 Effect of Torsional Root Spring Constant

Figs. (6-8) show the influence of the torsional root spring constant on the non-linear natural frequencies. For the first non-linear natural frequency Fig. (6) it can be noted that as torsional spring constant increase the non-linear natural frequency increase, but this increase becomes small for large values of torsional spring constant (a_4)

For higher modes (second and third) although the non-linear natural frequency will increase as torsional spring constant (a_4) increase Figs (7 and 8) but this increase is small, as we increase (a_4) from 10 to ∞ the non-dimensional natural frequency (ω) increased by a percentage of 4.6387% and 2.1561% at Ω=5 for the second and third mode respectively where it was 33.00% for the first mode.

5.4 The Effect of Attachment Angle

The effect of attachment angle on the first three frequencies is shown in Figs. (9-11)

For the first non-linear natural frequencies Fig. (9) as attachment angle increase the non-linear natural frequency will increase.

For higher modes (second and third) Figs. (10 and 11) the attachment angle hasn't a significant effect on non-linear natural frequencies. As we increase (η) from 0 to 70^o the non-dimensional natural frequency (ω) increased by a percentage of 0.6199% and 0.04896% at Ω=5 for the second and third mode respectively where it was 42.98% for the first mode.

5.5 Effect of Tip Mass

It can be seen from Figs. (12-14) That as the tip mass constant (a_2) increased the natural frequencies are decreased this is due to that as tip mass constant is increased, both the centrifugal force and mass of the system would increase. Increasing the centrifugal force will increase the non-linear natural frequencies, but the opposite effect of the tip mass constant is to increase the mass of the system (which is the dominant) and to decrease the natural frequencies.

As can be seen for Fig. (12) that for the first non-linear natural frequency the (MMS) and (SHB) fail to predict the behavior of rotating beam with tip mass constant greater than 0.2.

It can also be seen that at higher modes (Second and third) Figs. (13 and 14) the non-linear natural frequencies tend to converge to a common value for large values of a_2. As we increase (a_2) from 0 to 1 the non-dimensional natural frequency (ω) increased by a percentage of 32.1378% and 14.5963% at Ω=5 for the second and third mode respectively where it was increased by a percentage of 4.6307% and 0.7147% for second and third mode respectively as we increase a_2 from 1 to 100.

5.6 Effect of Rotary Inertia

It can be seen from Figs. (15-17) That as the rotary inertia constant (a_3) increased the natural frequencies are decreased.

Conclusions and Recommendations

6.1 Conclusions

In this study we have use the Method of Multiple Scales and the method of Harmonic Balance to find the non-linear natural frequencies of a rotating beam cantilever beam with tip mass and elastic root. The influence of the tip mass, the rotary inertia of the tip mass, the rotating speed, torsional spring constant, and hub radius ratio have been studied. It is shown that

1) The MMS and SHB give the same results approximately.

2) When increasing the hub radius ratio, the first three non-linear natural frequencies are increased.

3) When the tip mass parameter is increased, the first three non-linear natural frequencies are decreased.

4) When the torsional spring constant is increased the vibration frequencies increased.

5) When the rotary inertia constants are increased, the natural frequencies are decreased.

6) When attachment angle is increased the non-linear natural frequencies decreased.

6.2 Recommendations

It is recommended to carry out an experimental study to give more rigid results. It is also recommended to carryout second order MMS approximation and uses more harmonics in the assumed solution in the method of harmonic balance. It is also recommended to develop a new model, which take into account the coupled bending-bending vibration the effect of linear springs at the root of beam in addition to the torsional spring.

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الترددات الطبيعية اللاخطية لعارضة مرنة دوارة مثبتة على قاعدة مرنة

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ملخص

في هذه الدراسة قمنا باشتقاق نموذج لا خطي لعارضة دوارة مثبتة على قاعدة مرنة باستخدام معادلات لاغرانج وطريقة الصيغ الإفتراضية.

ثم قمنا باستخدام طريقة متعددة المقاييس وطريقة التوازن التوافقي لتحليل تأثير الكتلة الطرفية والقصور الدوراني والجذر المرن وزاوية الإتصال ونصف قطر السرة على الترددات الطبيعية اللاخطية .

أظهرت النتائج أنه بزيادة الكتلة الطرفية والقصور الدوراني فإن الترددات الطبيعية اللاخطية تقل، بينما بازدياد كل من نصف قطر السرة وزاوية الإتصال والجذر المرن فإن الترددات الطبيعية اللاخطية تزداد .